The symmetry between Electricity and Magnetism and the problem of the existence of a Magnetic Monopole

Georges Lochak
Fondation Louis de Broglie
23, quai de Conti 75006 Paris, France

1. Introduction

It seems fair to say that there are about as many physicists who consider the magnetic monopole as a monster hidden in the depths of the Loch Ness, as there are who regard this idea as so necessary for the beauty of nature, that God cannot possibly have failed to think about it. I belong of course to the latter species !

It is well known that the hypothesis of separated magnetic poles is an old one, but the present paper is neither devoted to its history nor to a comprehensive bibliography on the subject : there already exist several papers or books of this kind [1], [2], [3], [4]. Here, we shall quote only those papers that are useful for our purpose, which is to give arguments in favor of the hypothesis of magnetic monopoles, the possibility of their observation and the explanation of the fact that they were not yet observed with certainty.

Therefore, we shall not survey all aspects of the problem. In particular, although this is a commonly favoured point of view, there will be no further mention of a possible hyper-heavy monopole. Keeping away from G.U.T., we shall remain in the framework of electrodynamics. On the other hand, we shall not confine ourselves to symmetry arguments, but shall present a *wave equation* for a magnetic monopole, which parallels the Dirac equation for the electron. This equation describes a monopole quite different from the one which is usually considered, but it satisfies all the electrodynamical, mechanical and gauge properties commonly assumed at present. Needless to say, all these "properties" are conjectural and it is very likely that either there are no monopoles at all (God didn't think about it) or, if there is one, there must be a world of monopoles, just as large and diverse as the world of electrically charged particles.

2. At the beginning was symmetry

In 1894, one century ago, Pierre Curie wrote a paper on "Symmetry in Physical Phenomena" [5], where he put forward the idea of a general constructive role of symmetry in physics and emphasized the importance of *dissymmetry* in the appearance of phenomena. He described the *Curie groups* : a classification of physical invariance groups of limited objects in tridimensional space, in analogy with crystallographic groups, which are the invariance groups of an unlimited periodic medium. As an example, he described the symmetry of electromagnetic phenomena — and therefore of fields — entirely on the basis of *experiments,* without using Maxwell's equations (as, for instance, in [7]). He added a short paper [6] in which the possibility of "*free magnetic charges*" was shown as a consequence of the laws of symmetry of electromagnetic field[1].

There is a difference between electric and magnetic charges, which is a consequence of the fact that the electric field is a polar vector and the magnetic field is an axial one[2] : **E** has the symmetry of a radial

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[1] It is said in reference [2] that Curie "*suggests out of the blue that magnetic charge might exist*". It is no more "*out of the blue*" than all the predictions made in our century on the basis of symmetry, including the famous paper by Dirac himself on magnetic poles. Moreover, it was the first prediction of this kind.

[2] It is worth noticing that, despite the obvious difference between the two fields it is not so easy to prove experimentally which is polar and which is axial [5].

vector **r**, a velocity **v**, a linear momentum **p**, a force **F**, while **H** has the symmetry of the external product of two polar vectors, like **r**×**r'** or **r**×**p**.

As a consequence, consider the force exerted by each field on the corresponding charge :

$$\mathbf{F} = e\mathbf{E}, \qquad \mathbf{F} = g\mathbf{H} \tag{2.1}$$

If we assume that these law of force are P-invariant, the electric charge must be a scalar and the magnetic charge a pseudo-scalar : the image of an electric charge has the same sign, while the image of a north pole is a south pole and we find the following symmetry laws.
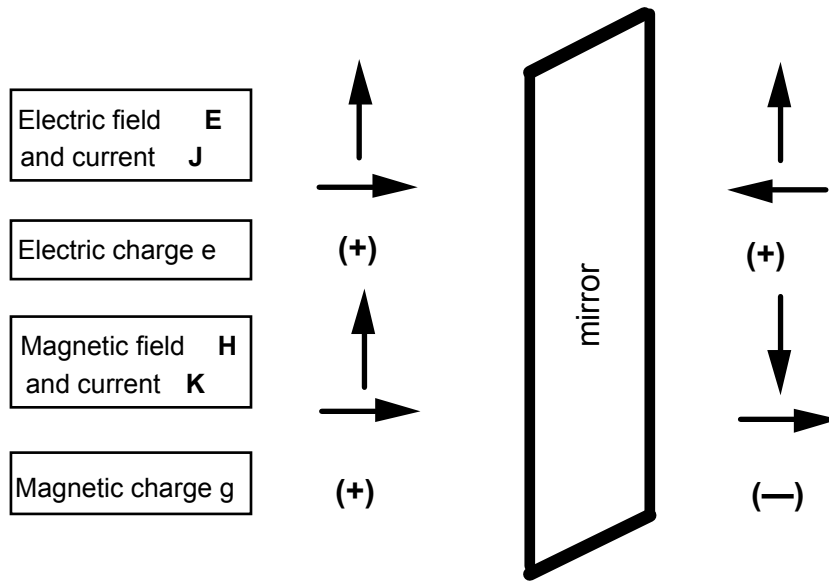

Fig. 1 Symmetry laws of electric and magnetic quantities

One can see on Fig.1 that, while the electric current is a polar vector, like the electric field, the magnetic current must be axial like the magnetic field, in virtue of the definitions :

$$\mathbf{J} = e\mathbf{v}, \qquad \mathbf{K} = g\mathbf{v} \tag{2.2}$$

It is astonishing to find a pseudoscalar physical constant g, because a physical constant has no tensorial variance : for instance, c does not vary as a velocity and h does not vary as an action or as a kinetic moment. Only physical quantities can have tensorial variances, not constants, and here, there is a confusion between the value of a constant and the variance of the corresponding physical quantity. We shall see that it is not so in quantum mechanics : the elementary magnetic charge will be a scalar, as it must be, but physical properties will be given by a *pseudo-scalar charge operator*. Magnetic current will be an axial vector different from (2.2).

In other words, Fig.1 that summarizes the work of Pierre Curie is true, but eq. (2.1) and (2.2) are not, and this is very important because a classical objection against the hypothesis of magnetic poles is that it is purely formal [2], [7], [8]. Actually, let us introduce densities of electric and magnetic currents and charges **J**, **K**, ρ, μ, in Maxwell's equations :

$$\text{curl } \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J} \; ; \; -\text{curl } \mathbf{E} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{K}$$

$$\text{div } \mathbf{E} = 4\pi\rho \ ; \ \text{div } \mathbf{H} = 4\pi\mu \tag{2.3}$$

This system is invariant under the transformation :

$$\mathbf{E} = \mathbf{E}' \cos \gamma + \mathbf{H}' \sin \gamma \ ; \ \mathbf{H} = -\mathbf{E}' \sin \gamma + \mathbf{H}' \cos \gamma$$

$$\rho = \rho' \cos \gamma + \mu' \sin \gamma \ ; \ \ \mu = -\rho' \sin \gamma + \mu' \cos \gamma$$

$$\mathbf{J} = \mathbf{J}' \cos \gamma + \mathbf{K}' \sin \gamma \ ; \ \ \mathbf{K} = -\mathbf{J}' \sin \gamma + \mathbf{K}' \cos \gamma \tag{2.4}$$

And the argument is that, by suitably choosing the angle γ,   one can arbitrarily eliminate magnetic (or electric) quantities. But this is true only if $\mathbf{J}$ and $\mathbf{K}$ are colinear, and it will not be true in our case, which invalidates the argument.

3. The Birkeland-Poincaré effect

In 1896, Birkeland introduced a straight magnet in a Crookes' tube and was puzzled by a convergence of the cathodic beam which does not depend on the orientation of the magnet [9]. Poincaré explained the effect by the action of a magnetic pole on the electric charges of the beam (these charges were only conjectured at that time) ; he showed that it is due to the action of only one pole of the magnet, and that, for symmetry reasons, it must be independent of the sign of the pole [10].
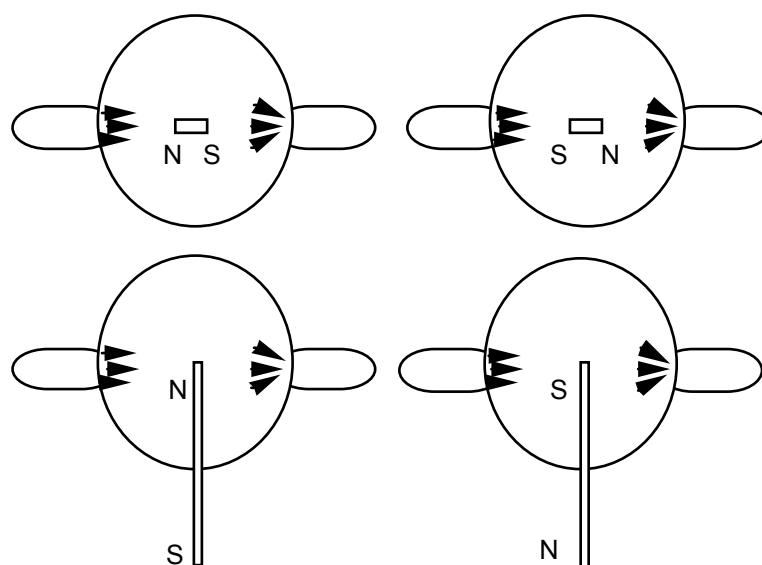

Fig. 2 The Birkeland-Poincaré effect. When a straight magnet
is introduced in a Crookes' tube, the cathodic rays converge
whatever the orientation of the magnet. Above : the cases
considered by Birkeland ; below : two cases corresponding to
the same description given by Poincaré.

In order to describe this effect, Poincaré wrote down the equation of motion of an electric charge in a coulombian magnetic field created by one end of the magnet. The magnetic field is :

$$\mathbf{H} = g \frac{1}{r^3} \mathbf{r}$$

(3.1)

where g is the magnetic charge, and, from the expression of the Lorentz force, we find the Poincaré equation :

$$\frac{d^2\mathbf{r}}{dt^2} = \lambda \frac{1}{r^3} \frac{d\mathbf{r}}{dt} \times \mathbf{r} \; ; \quad \lambda = \frac{eg}{mc}$$

(3.2)

where e and m are the electric charge and the mass of the electron.

Poincaré found the following integrals of motion, where A, B, C, Λ, are arbirary constants :

$$\mathbf{r}^2 = C\,t^2 + 2B\,t + A \; ; \; \left(\frac{d\mathbf{r}}{dt}\right)^2 = C$$

(3.3)

$$\mathbf{r} \times \frac{d\mathbf{r}}{dt} + \lambda \frac{\mathbf{r}}{r} = \Lambda$$

(3.4)

He obtained from eq. (3.4) :

$$\Lambda \cdot \mathbf{r} = \lambda r \; ; \; \frac{d^2\mathbf{r}}{dt^2} \cdot \mathbf{r} = \frac{d^2\mathbf{r}}{dt^2} \cdot \frac{d\mathbf{r}}{dt} = 0$$

(3.5)

which says that **r** describes an axially symmetric cone — the *Poincaré cone* — and that the acceleration is perpendicular to its surface, so that **r** follows a *geodesic line* .

If the cathodic rays are emitted far away from the magnetic pole with a velocity V parallel to the z axis, they will have an asymptote which obeys the equations :

$$x = x_0 \; ; \; y = y_0$$

(3.6)

And we find from (3.3) and (3.4) :

$$C = V^2 \; ; \; \Lambda = \left\{ y_0 V, - x_0 V, \lambda \right\}$$

(3.7)

The z axis is thus a generating line of the Poincaré cone and the half angle Θ' at the vertex is given by :

$$\sin \Theta' = \frac{V}{\lambda} \sqrt{x_0^2 + y_0^2}$$

(3.8)

Now, the cathodic ray that becomes, after the emission, a geodesic line rotating along the cone, crosses the z axis at distances from the origin given by

$$\frac{\sqrt{x_0^2 + y_0^2}}{\sin \phi} \; ; \; \frac{\sqrt{x_0^2 + y_0^2}}{\sin 2\phi} \; ; \; \frac{\sqrt{x_0^2 + y_0^2}}{\sin 3\phi} \; ;\dots \;\; \phi = 2\pi \sin \Theta'$$

(3.9)

Therefore, if the emitting cathode is a small disc of radius $\sqrt{x_0^2 + y_0^2}$ orthogonal to the z axis, and if the position of the magnetic pole is such that one of these points is on the surface of the tube, there will be a concentration of the electrons emitted by the periphery of the cathode and even, approximately, of those coming from the whole disc : this is the *focusing* effect observed by Birkeland.

This is an important result because, although the existence of magnetic monopoles as particles is not yet proved (at least, we are not sure of that), the Poincaré equation (3.2) and the integral of motion (3.4) are experimentally verified.

In eq. (3.4), the first term is clearly the orbital momentum of the electron with respect to the magnetic pole. The second term was later interpreted by J.J. Thomson (see [7], [11]) who showed that :

$$\frac{eg}{c}\frac{\mathbf{r}}{r} = \frac{1}{4\pi c}\int_{-} \mathbf{x} \times (\mathbf{E} \times \mathbf{H})\, d^3x$$

(3.10)

Thus, with the value of λ given in eq. (3.2), the second term of the Poincaré integral is equal to the electromagnetic momentum and eq. (3.4) gives the *constant total angular momentum* **J**=m**Λ**. The presence of a non vanishing electromagnetic angular momentum is due to the axial character of the magnetic field created by a magnetic pole and acting on the electric charge.

Let us add a remark about symmetry [12] : the Poincaré cone is enveloped by a vector **r** which is the *symmetry axis* of the system formed by the electric and the magnetic charge, and this axis rotates (with a constant angle Θ') around the constant *angular momentum* **J**=m**Λ**. But this is exactly the definition of the *Poinsot cone* associated to a symmetric top [13].

The Poincaré cone is nothing but the Poinsot cone of a symmetrical top, which is not surprising because the system formed by an electric and a magnetic charge is axisymmetric and is rotating around a fixed point with a constant total angular momentum. Such a system must have the angular properties of a top, but with a different radial motion because the it is not rigid (the motion along the geodesic lines of the cone has nothing to do with a top).

Introducing the following definition with two obvious properties :

$$\mathbf{L} = \mathbf{r} \times \frac{d\mathbf{r}}{dt} \; ; \; \mathbf{L} \cdot \frac{\mathbf{r}}{r} = 0 \; ; \; \mathbf{\Lambda} \cdot \frac{\mathbf{r}}{r} = 0$$

(3.11)

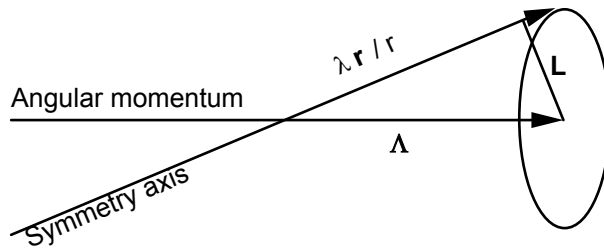all that was said can be summarized in the following figure :

Fig. 3 The generation of the Poincaré (or Poinsot) cone
and the decomposition of the total momentum.

Of course, all these results are true for a magnetic charge in a coulombian electric field : we shall see that this will be true in our case and that our equation for a magnetic monopole will give, at the classical limit, the Poincaré equation.

4. Forces and potentials for a magnetic pole

Owing to the second formula (2.1), we can write the equation of motion of a monopole in a particular system where the external field reduces to its magnetic part :

$$\frac{d\,\mathbf{p}_0}{dt_0} = g\,\mathbf{H}_0$$

(4.1)

$\mathbf{p}_0$, t_0 and $\mathbf{H}_0$ are the momentum, time and magnetic field in this system. The Lorentz transformation of the electromagnetic field is :

$$\mathbf{E}_0 = \frac{\mathbf{E} + v/c\,\mathbf{H}}{\sqrt{1-(v/c)^2}}\;;\;\;\mathbf{H}_0 = \frac{\mathbf{H} - 1/c\,\mathbf{v}\times\mathbf{E}}{\sqrt{1-(v/c)^2}}$$

(4.2)

Therefore, the general form of (4.1) reads :

$$\frac{d\,\mathbf{p}}{d\tau} = g\,\frac{\mathbf{H} - 1/c\,\mathbf{v}\times\mathbf{E}}{\sqrt{1-(v/c)^2}}$$

(4.3)

where $d\tau = dt\sqrt{1-(v/c)^2}$ is the differential of the proper time so that (4,3) can be writen :

$$\frac{d\,\mathbf{p}}{dt} = g\left(\mathbf{H} - 1/c\,\mathbf{v}\times\mathbf{E}\right)$$

(4.4)

The right-hand side of eq. (4.4) is the Lorentz force acting on a magnetic pole, with a minus sign in front of $\mathbf{E}$ instead of the plus sign occuring in front of $\mathbf{H}$ in the electric case.

Now, we go back to the Maxwell equations (2.3) with magnetic current and charge densities and introduce relativistic coordinates :

$$x^\alpha = \left\{x^1, x^2, x^2, x^4\right\} = \left\{x, y, z, ict\right\}$$

(4.5)

In a covariant form, eq. (2.3) becomes :

$$\check{Z}^\beta F_{\alpha\beta} = \frac{4\pi}{c}\,J_\alpha\;;\;\;J_\alpha = \left(\mathbf{J}, i\rho c\right)$$
$$\check{Z}^\beta \overline{F}_{\alpha\beta} = \frac{4\pi}{c}\,K_\alpha\;;\;\;iK_\alpha = \left(\mathbf{K}, i\mu c\right)$$

(4.6)

where the $i = \sqrt{-1}$ in front of K_α is due to the axial character of $\mathbf{K}$; we have the relation of duality:

$$\overline{F}_{\alpha\beta} = \frac{i}{2}\,\varepsilon_{\alpha\beta\gamma\delta}\,F^{\gamma\delta}\quad(\varepsilon_{\alpha\beta\gamma\delta}\text{ - antisymmetric})$$

(4.7)

It is clear that we cannot define the field by a Lorentz *polar* potential only because :

$$F_{\alpha\beta} = \check{Z}_\alpha A_\beta - \check{Z}_\beta A_\alpha \to \check{Z}^\beta \overline{F}_{\alpha\beta} = 0$$

(4,8)

Then, we must introduce a new potential B_α such that :

$$F_{\alpha\beta} = \check{Z}_\alpha A_\beta - \check{Z}_\beta A_\alpha + \overline{\check{Z}_\alpha B_\beta - \check{Z}_\beta B_\alpha}$$

(4.9)

Both right-hand terms in eq. (4.9) must have the same variance. Hence, B_α is a pseudo-potential, i.e. the dual of an antisymmetric tensor of rank three :

$$B_\alpha = \overline{C_{\beta\gamma\delta}} = \frac{1}{3!}\, \varepsilon_{\alpha\beta\gamma\delta} C^{\beta\gamma\delta} \tag{4.10}$$

In terms of ordinary coordinates, we have :

$$A_\alpha = (\mathbf{A}, iV) \;;\quad iB_\alpha = (\mathbf{B}, iW) \tag{4.11}$$

where $\mathbf{B}$ is an axial vector. The fields are defined as :

$$\mathbf{E} = -\nabla V - \frac{1}{c}\frac{\check{Z}\mathbf{A}}{\check{Z}t} + \mathrm{curl}\ \mathbf{B}$$

$$\mathbf{H} = \mathrm{rot}\ \mathbf{A} + \nabla W + \frac{1}{c}\frac{\check{Z}\mathbf{B}}{\check{Z}t} \tag{4.12}$$

Actually, we shall not consider "dyons" with electric and magnetic charges, but "true" magnetic poles with a magnetic charge only, so that formulae (4.9) and (4.12) reduce to :

$$F_{\alpha\beta} = \check{Z}_\alpha B_\beta - \check{Z}_\beta B_\alpha \ \rightarrow\ \mathbf{E} = \mathrm{curl}\ \mathbf{B}\ ;\ \mathbf{H} = \nabla W + \frac{1}{c}\frac{\check{Z}\mathbf{B}}{\check{Z}t} \tag{4.13}$$

The last formulae were derived by de Broglie from his theory of light [14] ; they were related to the magnetic monopole by Cabibbo and Ferrari [15].

5. Dirac strings

In a celebrated paper of 1931, Dirac raised a fundamental problem about the interaction between electric and magnetic charges i. e. either the motion of an electric charge around a fixed monopole or conversely the motion of a monopole around a fixed electric charge, [16], [17], [18], [19]. Let us choose, as Dirac did, the motion of an electric charge in the magnetic coulombian field $\mathbf{H}$ generated by a fixed monopole with charge g. $\mathbf{H}$ is thus defined by a vector potential $\mathbf{A}$ such that:

$$\mathrm{curl}\ \mathbf{A} = g\,\frac{\mathbf{r}}{r^3} \tag{5.1}$$

It is clear that there is no continuous and uniform solution $\mathbf{A}$ of this differential equation because if we consider a surface Σ bounded by a loop Λ, we find according to Stokes' theorem :

$$\int_\Sigma \mathbf{H}.d\mathbf{S} = \int_\Sigma \mathrm{curl}\ \mathbf{A}.d\mathbf{S} = \int_\Lambda \mathbf{A}.d\mathbf{l} = g\int_\Sigma \frac{\mathbf{r}}{r^3}.d\mathbf{S} = g\int_\Sigma d\Omega \tag{5.2}$$

where $d\mathbf{S}$, $d\mathbf{l}$ and $d\Omega$ are elements of surface, length and solid angle respectively. Now, if the loop is shrinked to a point, while the pole remains inside the closed surface Σ, we get :

$$\int_{\Lambda\to 0} \mathbf{A}.d\mathbf{l} = g\int_\Sigma d\Omega = 4\pi g \tag{5.3}$$

This equality is impossible for a continuous potential **A** because then the first integral vanishes. There must be a singular line somewhere around which the loop shrinks. Now, whatever the wave equation, the minimal coupling is given by covariant derivatives :

$$\nabla - i\,\frac{e}{hc}\,\mathbf{A}$$

(5.4)

Dirac introduced in the wave function ψ a non integrable (non univalent) phase γ, defining a new wave function :

$$\Psi = e^{i\gamma}\,\psi$$

(5.5)

If we apply the preceding operator, we know that the introduction of this phase γ is equivalent to the introduction of a new potential by a change of electromagnetic gauge :

$$\left(\nabla - i\,\frac{e}{hc}\,\mathbf{A}\right)\Psi = e^{i\gamma}\left(\nabla + i\nabla\gamma - i\,\frac{e}{hc}\,\mathbf{A}\right)\psi$$

(5.6)

We can identify the new potential with the gradient of γ, but the phase factor $e^{i\gamma}$ is admissible only if the variation of γ around a closed loop is equal to a multiple of 2π. Then, we must have :

$$\frac{e}{hc}\oint_{\Lambda\to 0}\mathbf{A}.d\mathbf{l} = \int_{\Lambda\to 0}\nabla\gamma.d\mathbf{l} = \left(\Delta\gamma\right)_{\text{loop}} = 2\pi n$$

(5.7)

Comparing eq. (5.3) and (5.7), we find the famous *Dirac condition* between electric and magnetic charges :

$$\frac{eg}{hc} = \frac{n}{2}$$

(5.8)

It is interesting to confirm this result on a solution of the eq. (5.1). Dirac chose the following solution :

$$A_x = \frac{g}{r}\frac{-y}{r+z}, \quad A_y = \frac{g}{r}\frac{x}{r+z}, \quad A_z = 0, \quad r = \sqrt{x^2+y^2+z^2}$$

(5.9)

In polar coordinates :

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\cos\varphi, \quad z = r\cos\theta$$

(5.10)

Eq. (5.9) becomes :

$$A_x = -\frac{g}{r}\tan\frac{\theta}{2}\sin\varphi, \quad A_y = \frac{g}{r}\tan\frac{\theta}{2}\cos\varphi, \quad A_z = 0$$

(5.11)

There is a nodal line which goes from $x = 0$ to ∞, for $\theta = \pi$, and the Dirac condition is easily found if we compute the curvilinear integral (5.7) around this line for $\theta = \pi - \varepsilon$ and let ε go to 0. We must have :

$$\frac{e}{hc}\oint_{\Lambda \to 0} \mathbf{A}.d\mathbf{l} = \frac{eg}{hc}\oint_{\theta = \pi - \varepsilon \, , \, \varepsilon \to 0} \frac{1}{r}\tan\frac{\theta}{2} \; r\sin\theta \; d\varphi = 2\pi n \tag{5.12}$$

Therefore :

$$\frac{eg}{hc}\oint_{\varepsilon \to 0} \frac{\sin\varepsilon}{\tan\frac{\varepsilon}{2}} \; d\varphi = \frac{eg}{hc}2\times 2\pi = 2\pi n \tag{5.13}$$

We see that the factor 2 comes from $\varepsilon/2$ in the tangent and we could conclude that it is related to the fact that the nodal line begins at $r = 0$. But this is wrong because the solution (5.9) or (5.11) chosen by Dirac depends on an arbitrary gauge, and his choice is not actually very good because *this potential has no definite parity*. Moreover, it must be stressed that with a *polar vector* **A**, the vector curl **A** is axial, so that eq. (5,1) would be admissible only with a *pseudo scalar constant* g, against which we have already objected. In the following, we shall find a wave equation for a monopole in an electromagnetic field and our potential will not be **A** but **B**, solution of the following equation, where e is a scalar :

$$\text{curl } \mathbf{B} = e\frac{\mathbf{r}}{r^3} \tag{5.14}$$

We know that **B** is an axial vector and it is evident on (5.14) because curl **B** must be polar, like **r**. *Mutadis mutandis* Dirac's reasoning presented above will be true, but we shall choose another solution of (5.14) which is axial :

$$B_x = \frac{e}{r}\frac{yz}{x^2 + y^2} \; , \quad B_x = \frac{e}{r}\frac{-xz}{x^2 + y^2} \; , \quad B_z = 0 \; , \quad r = \sqrt{x^2+y^2+z^2} \tag{5.15}$$

Or, in polar coordinates :

$$B_x = \frac{e}{r}\frac{\sin\varphi}{\tan\theta} \; , \quad B_y = \frac{e}{r}\frac{-\cos\varphi}{\tan\theta} \; , \quad B_z = 0 \tag{5.16}$$

The Dirac-like solution is, in this case :

$$B'_x = \frac{e}{r}\frac{-y}{r+z} \; , \quad B'_y = \frac{e}{r}\frac{x}{r+z} \; , \quad B'_z = 0 \tag{5.17}$$

And we have the gauge difference :

$$\mathbf{B}' - \mathbf{B} = \nabla \; \text{Arctan}\frac{y}{x} \tag{5.18}$$

Using solution (5.15) or (5.16) in Dirac's proof of relation (5.8), the singular line goes now from $-\infty$ to ∞, instead of from 0 to ∞, and equality (5,13) becomes :

$$2 \times \frac{eg}{hc} \int_{\varepsilon \to 0} \frac{\sin \varepsilon}{\tan \varepsilon} \, d\varphi = 2 \times \frac{eg}{hc} 2\pi = 2\pi n$$

(5.19)

Now, the factor 2 comes no more as it did above from $\tan \varepsilon/2$ but from the fact that the singular line pierces the sphere in *two points*. Therefore, the factor n/2 in the Dirac condition (5,8) was not at all related to the fact that the singular line began in r = 0. Further, we shall give another proof of Dirac's relation and we shall see that *the factor n/2 is related to the double connexity of the rotation group*.

According to (5.8), if we choose the charge e of the electron as a unit electric charge, the magnetic charge is quantized. For n=1, we obtain the unit magnetic charge as a function of the electron charge and of the fine structure constant :

$$g_0 = \frac{hc}{2e^2} e = \frac{1}{2\alpha} e \cong \frac{137}{2} e = 68{,}5 \, e$$

(5.20)

This is an enormous charge, of the order of the electric charge of a nucleus in the region of lantanides, beyond the middle of Mendeleïev's classification. This means that a monopole interacts strongly with atoms and possesses strong ionizing properties. It seems that a light monopole would be continually deviated from its inertial trajectory. Actually, the problem of the interaction cross section between electric and magnetic particles is difficult and not free from some ambiguities or ad hoc hypotheses which we shall not discuss in this paper (see [1], [20], [21], [22], [23], [24]).

It is noteworthy, for future questions raised by experiment, that Dirac's condition (5,8) is based on general assumptions from quantum mechanics (uniformity of phase), electromagnetism (fields expressed by potentials) and on the postulate of gauge invariance[1]. Until now, there has been unfortunately only one observation, by Blas Cabrera, of a possible magnetic monopole, with a magnetic charge obeying the Dirac condition : a jump of the quantized magnetic flux of a superconducting loop, corresponding to one Dirac unit (5.20) of magnetic charge [25]. But we cannot ignore the fact that a series of experiments was performed in the thirties by Ehrenhaft [26], later by Schedling [27] and Ferber [28], and recently by Mihaïlov [29], [30], [31], [32], [33]. Thousands of observations of strongly illuminated ferromagnetic aerosols in various electric and magnetic fiels, showed a migration of microparticles which could not be interpreted, until now, other than as a migration of magnetic monopoles (several other interpretations were attempted, but seem to be wrong).

Nevertheless, these phenomena are not really understood. They give a whole range of values of magnetic charge, and most of them *violate the Dirac condition*. While the unit given by (5,20) is about $3{,}28 \times 10^{-8}$ gauss cm^2, the measured charge is between 10^{-13} to 10^{-11} gauss cm^2. Mikhaïlov gave an empirical formula that fits many experimental results [31] :

$$g = \frac{\alpha}{6} e \Rightarrow g = \frac{1}{3} \alpha^2 g_0$$

(5.21)

α is the fine structure constant and g_0 the Dirac unit charge (5.20).

In 1948, Dirac [17] quoted two Ehrenhaft's papers [34]. He did not allude directly to the discrepancy concerning the charge, but he remarked that the value he had predicted for the smallest magnetic charge led him to the conclusion that the creation of a magnetic pole requires an enormous amount of energy in order to separate a magnetic doublet[2]. But Ehrenhaft did not use high energies, so that Dirac simply said : "*this is*

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[1]It may be noticed that Dirac's result on monopole was perhaps the first one which was based on the postulate of gauge invariance.

[2] We shall see later that Dirac's conclusion is not at all obvious because the separation of a doublet is not the unique way to produce a monopole. Perhaps it is not the way at all.

not a confirmation of the present theory". But, of course, he didn't assert that the experiments were impossible for theoretical reasons !

Einstein was informed of these experiments and of Ehrenhaft's conclusion that, perhaps, even the fundamental electric charge is not a universal constant. In 1939, in a letter to Langevin, Einstein made a brief but remarkable comment and recommended Ehrenhaft for the next *Conseil Solvay* (that actually met after the war) :

"*I know that Ehrenhaft' reputation is bad in the circles of physicists because of his persistency, based on numerous researches, to deny the constancy of the elementary quantum of electricity. Nevertheless, it must be said in his favour, that he himself has created a large part of the method in question, and that the results of his experiments were not weakened by more reliable results of measures performed in this domain, but rather by general reflections, based on results obtained in a quite different domain.*" [35]

A beautiful example of Einstein's open mind and of his humility before experimental physics.

6. A spinor wave equation for a magnetic monopole.

Now, we shall find a wave equation for a magnetic monopole. It will be shown that the famous Dirac equation for the electron admits not only one local gauge invariance but two — and only two. As it is known, the first invariance (the ordinary phase invariance) corresponds to an electric charge ; the second one corresponds to a magnetic monopole. The new spinorial equation so obtained describes, in quantum terms, the Curie symmetry laws, and we shall see that new symmetries are involved, that can be given only in quantum mechanics, without any classical equivalence : especially the correct definition of an antimonopole.

a) Scalar gauge and chiral gauge in the Dirac equation.

Let us write the Dirac equation without external field :

$$\gamma_\mu \check{Z}_\mu \psi + \frac{m_0 c}{h} \psi = 0 \tag{6.1}$$

where $x_\mu = \{x_k ; ict\}$ and γ_μ are matrices defined in terms of Pauli matrices s_k as :

$$\gamma_k = i \begin{pmatrix} 0 & s_k \\ -s_k & 0 \end{pmatrix} ; \ k = 1, 2, 3 ; \ \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} ; \ \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{6.2}$$

Consider a global gauge transformation where Γ is a Hermitian matrix and θ a constant phase :

$$\psi \rightarrow e^{i\Gamma\theta} \psi \tag{6.3}$$

Eq. (6.1) becomes :

$$\left(\gamma_\mu e^{i\Gamma\theta}\gamma_\mu\right)\gamma_\mu\psi + \frac{m_0 c}{h} e^{i\Gamma\theta} \psi = 0 \tag{6.4}$$

Let us now develope Γ on the following Clifford algebra basis built on the γ_μ matrices :

$$\Gamma = \sum_{N=1}^{16} a_N \Gamma_N ; \ \Gamma_N = \left\{ I, \gamma_\mu, \gamma_{[\mu}\gamma_{\nu]}, \gamma_{[\lambda}\gamma_\mu\gamma_{\nu]}, \gamma_5 \right\} \tag{6.5}$$

Owing to the commutation rules of γ_μ we have the following relations for any matrix Γ_N [36] (the sign varies with μ and N) :

$$\gamma_\mu \, \Gamma_N \, \gamma_\mu = \pm \, \Gamma_N \tag{6.6}$$

Hence we find from eq. (6.4) :

$$\gamma_\mu \, e^{i\Gamma\theta} \, \gamma_\mu = \exp\left(i\theta \sum_{N=1}^{16} \pm a_N \, \gamma_\mu \, \Gamma_N \gamma_\mu \right) = \exp\left(i\theta \sum_{N=1}^{16} \pm a_N \, \Gamma_N \right) \tag{6.7}$$

A necessary condition for the gauge invariance of eq. (6.1) is that the factor $\left(\gamma_\mu \, e^{i\Gamma\theta}\gamma_\mu \right)$ in (6.4) does not depend on μ and *this is possible with two and only two matrices* Γ_N : I and γ_5 because the first commutes and the second anticommutes with all the γ_μ . Therefore :

$$\Gamma = a_1 I + a_5 \, \gamma_5 \;\; ; \;\; a_1 \, , \;\; a_5 = \text{Const.} \tag{6.8}$$

The first term gives the *ordinary phase invariance* that implies the *conservation of electricity* :

$$\Gamma = I \; ; \;\; \psi \rightarrow e^{i\theta} \, \psi \tag{6.9}$$

For reasons to be explained shortly, the second term will be called the *chiral invariance* :

$$\Gamma = \gamma_5 \; ; \;\; \psi \rightarrow e^{i\gamma_5\theta} \, \psi \tag{6.10}$$

We shall see that this invariance implies the *conservation of magnetism*. But these two gauges are quite different because the first one is valid for every value of m_0 in eq. (6.1), so that the conservation of electricity is universal in quantum mechanics, while the second one is valid only for $m_0 = 0$ in (6.1) because of the anticommutation of γ_5 and γ_μ, so that the conservation of magnetism is not as strong as the conservation of electricity ; it is broken by a linear mass term. Despite this difference, there is a symmetry between the two gauges, owing to two kinds of relations.

At first, let us recall that the Dirac spinor and the Clifford algebra basis (6.5) define 16 tensorial quantities : a scalar, a polar vector, an antisymmetric tensor of rank two, an antisymmetric tensor of rank three (an axial vector) and an antisymmetric tensor of rank four (a pseudo-scalar) :

$$\Omega_1 = \overline{\psi} \, \psi \; ; \; J_\mu = i \, \overline{\psi} \, \gamma_\mu \, \psi \; ; \; M_{\mu\nu} = i \, \overline{\psi} \, \gamma_\mu \, \gamma_\nu \, \psi \; ; \; \Sigma_\mu = i \, \overline{\psi} \, \gamma_\mu \, \gamma_5 \, \psi \; ; \; \Omega_2 = - \, i \, \overline{\psi} \, \gamma_5 \, \psi$$
$$\left(\overline{\psi} = \psi^+ \gamma_4 \; ; \;\; \psi^+ = \text{hermitean conjugated} \right)$$
$$\tag{6.11}$$

Secondly, when Ω_1 and Ω_2 do not simultaneously vanish, the Dirac spinor may be written as follows [37], [38], [39] :

$$\psi = \rho \; e^{i \, \gamma_5 A/2} \, U \, \psi_0 \tag{6.12}$$

ψ_0 is a constant spinor, U a general Lorentz transformation, A the *pseudo-scalar angle* of Yvon-Takabayasi :

$$\tan A = \frac{\Omega_2}{\Omega_1} \tag{6.13}$$

The amplitude ρ is :

$$\rho = \sqrt{\Omega_1{}^2 + \Omega_2{}^2} \tag{6.14}$$

Now, the proper rotation Euler angle φ (included in U) defines a *scalar phase* $\varphi/2$ of the spinor ψ, which is canonically conjugate (with respect to a *classical* Poisson bracket) to the fourth component of the *polar vector* J_μ [37], [38], [39] :

$$\left[\frac{\varphi}{2}, J_4 \right] = \delta\left(\mathbf{r} - \mathbf{r}' \right) \tag{6.15}$$

In analogy with (6.15), the pseudo-scalar phase $A/2$ is conjugate to the fourth component of the *axial vector* Σ_μ [37], [38], [39] :

$$\left[\frac{A}{2}, \Sigma_4 \right] = \delta\left(\mathbf{r} - \mathbf{r}' \right) \tag{6.16}$$

In Dirac's therory of the electron, J_4 is a density of *electricity* associated with the phase invariance and the space part $\mathbf{J}$ of J_μ is a density of electric current. In the same way, Σ_4 is a density associated to the chiral invariance and the space part $\mathbf{\Sigma}$ of Σ_μ is a density of current. They will be densities of charge and current of *magnetism*.

Nevertheless, among the differences between the two gauges, and apart from the fact that J_μ is *polar* and Σ_μ *axial*, there is the important property that J_μ is *time-like* while Σ_μ is *space-like* because of the Darwin - De Broglie equalities :

$$- J_\mu J_\mu = \Sigma_\mu \Sigma_\mu = \Omega_1{}^2 + \Omega_2{}^2 \tag{6.17}$$

The fact that J_μ (interpreted as a current of electricity and probability) is time-like is very important because this property is equivalent to the existence of a rest-frame. At first glance, a space-like magnetic current Σ_μ looks unacceptable but we shall see that it is not so.

The fact that A is a pseudo-scalar is easy to prove using the transformations :

$$P : \psi \rightarrow \gamma_4 \psi \; ; \; T : \psi \rightarrow \gamma_1 \gamma_2 \gamma_3 \psi \; ; \; C : \psi \rightarrow \gamma_2 \psi^* \tag{6.18}$$

With the definitions (6.11), this implies :

$$P : \Omega_1 \rightarrow \Omega_1 \; ; \; \Omega_2 \rightarrow -\Omega_2$$
$$T : \Omega_1 \rightarrow -\Omega_1 \; ; \; \Omega_2 \rightarrow \Omega_2$$
$$C : \Omega_1 \rightarrow -\Omega_1 \; ; \; \Omega_2 \rightarrow -\Omega_2 \tag{6.19}$$

Then, definition (6.13) shows that A is a relativistic pseudo-invariant[1]. Now, we can give a more geometrical interpretation of the chiral gauge transformation, making use of the equalities :

$$\Omega_1 = \rho \cos A \; ; \; \Omega_2 = \rho \sin A \; ; \; \rho = \sqrt{\Omega_1{}^2 + \Omega_2{}^2} \qquad (6.20)$$

which are P, T and C-invariant provided that we admit the following conditions on A :

$$P : A \to -A \; ; \; T : A \to -A - \pi \; ; \; C : A \to A + \pi \qquad (6.21)$$

Now, consider a chiral gauge transformation, slightly modified with respect to the eq. (6,10) :

$$\psi' = e^{i\Gamma\theta/2} \, \psi \qquad (6.22)$$

Using definition (6,11) of Ω_1 and Ω_2, we get :

$$\begin{pmatrix} \Omega_1' \\ \Omega_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} \qquad (6.23)$$

The chiral gauge transformation is thus a θ rotation in the plane $\{\Omega_1, \Omega_2\}$ (while the rotation of the spinor was $\theta/2$). Making use of (6.13), we verify that θ represents a phase shift of the angle A :

$$A' = A + \theta \qquad (6.24)$$

Naturally, θ is a *relativistic pseudo-scalar*, like A.

b) The wave equation.

We know that, introducing the ordinary local gauge (5.5) in the Dirac equation (6.1), we find the minimal electric coupling and the covariant derivatives (5.4) in terms of Lorentz potentials, that give the equation of an electric particle in an electromagnetic field.

Now, consider the Dirac equation with $m_0 = 0$:

$$\gamma_\mu \tilde{Z}_\mu \psi = 0 \qquad (6.25)$$

and the chiral gauge (6.10), replacing the abstract pseudo-scalar angle θ with a (pseudo-scalar) phase ϕ with physical coefficients :

$$\psi \to \exp\left(i \, \frac{g}{hc} \, \gamma_5 \, \phi\right) \psi \; ; \; B_\mu \to B_\mu + i \, \partial_\mu \phi \qquad (6.26)$$

---

[1] It must be noted that Ω_1 and Ω_2 are respectively a scalar and a pseudo-scalar, but not in the four-dimensional space-time, only the three-dimensional space. The sign of Ω_1 remains unchanged by P, while the sign of Ω_2 is changed. Conversely, the sign of Ω_1 is changed by T, while the sign of Ω_2 remains unchanged.

g will be a *scalar* magnetic charge : the pseudo-scalar character of magnetism is related to a pseudo-scalar *magnetic charge operator* G which is at the origin of all the differences between the classical and the quantum theory of magnetic monopoles :

$$G = g \, \gamma_5 \tag{6.27}$$

As ϕ is a pseudo-scalar, the electromagnetic potential cannot be the Lorentz polar vector A_μ, but the axial potential B_μ, defined by (4.10), (4.11) and which have the variance of $\partial_\mu \phi$. The covariant derivatives are now (the absence of i in front of g is due to the axiality of B_μ) :

$$\nabla_\mu = \partial_\mu - \frac{g}{hc} \gamma_5 B_\mu \tag{6.28}$$

The equation of the magnetic monopole is thus [39], [40]:

$$\gamma_\mu \left(\partial_\mu - \frac{g}{hc} \gamma_5 B_\mu \right) \psi = 0 \tag{6.29}$$

The justification of this equation will be given by its symmetry properties and by the motion in a central electric field.

7. Symmetries of the wave equation.

a) Gauge invariance.

By definition, eq. (6.29) is invariant with respect to the chiral gauge transformation (6.23). This entails the conservation of the *axial current* that will play the role of a magnetic current :

$$\partial_\mu K_\mu = 0 \; ; \quad K_\mu = g \, \Sigma_\mu = i \, g \, \overline{\psi} \, \gamma_\mu \, \gamma_5 \, \psi \tag{7.1}$$

It must be noticed that this magnetic current is not parallel to the electric current and that its pseudo-tensorial variance is in accordance with the Curie laws. The question of its space-like character will become clear a little further. The same expression for the magnetic current was previously suggested by Salam [41] for reasons of symmetry, but here, it is deduced from a wave equation and a gauge condition.

b) CPT.

It is easy to prove that our wave equation is C, P and T invariant, i.e. invariant under the transformations :

$$P : x_k \rightarrow -x_k \; ; \; x_4 \rightarrow x_4 \; ; \; B_k \rightarrow B_k \; ; \; B_4 \rightarrow -B_4 \; ; \; \psi \rightarrow \gamma_4 \psi$$

$$T : x_k \rightarrow x_k \; ; \; x_4 \rightarrow -x_4 \; ; \; B_k \rightarrow -B_k \; ; \; B_4 \rightarrow B_4 \; ; \; \psi \rightarrow \gamma_1 \gamma_2 \gamma_3 \psi$$

$$C : g \rightarrow g \; ; \; \psi \rightarrow \gamma_2 \psi^* \quad \left(\psi^* = \text{compl. conj.} \right) \tag{7.2}$$

In this formulae, the most important point is that *the charge conjugation does not change the sign of the magnetic constant of charge*. In the next section, we shall see what exactly charge conjugation means, but we can already assert that two conjugated monopoles have the same charge constant and that two monopoles with opposite charges are not charge conjugated : changing g in $-$g in eq. (6.29), we find a new equation which is not unitary equivalent to the original one.

Therefore, we cannot create or annihilate pairs of monopoles with opposite charges g and $-g$, in the way pairs of electric charges are created or annihilated. The properties of charge conjugation of eq. (6.29) show that there is no danger of an infinite polarization of vacuum which could occur from the zero mass of our monopole and it shows that *one cannot invoke the hypothesis of great masses* to explain the rarity of monopoles or, at least, the difficulty to observe them. The fact that chiral invariance and conservation of magnetism are easily broken shows that, more probably, monopoles are abundant in nature and that the difficulty of the isolation of one of them is not a question of energy.

But what are conjugated monopoles ?

8. Weyl's representation. Two-component theory.

The Weyl representation is the one that diagonalizes γ_5 and, thus, the charge operator G. The transformation is :

$$\psi \to U\,\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix};\ \ U = U^{-1} = \frac{1}{\sqrt{2}}\left(\gamma_4 + \gamma_5\right)$$

(8.1)

where ξ and η are two-component spinors ; and we have :

$$U\,G\,U^{-1} = U\,g\,\gamma_5\,U^{-1} = g\,\gamma_4 = \begin{pmatrix} g & 0 \\ 0 & -g \end{pmatrix}$$

(8.2)

Applying eq. (8.2) to ψ given by (8.1), we see that ξ and η are eigenstates of G, corresponding to the eigenvalues g and $-g$:

$$U\,G\,U^{-1}\begin{pmatrix} \xi \\ 0 \end{pmatrix} = g\begin{pmatrix} \xi \\ 0 \end{pmatrix};\ \ U\,G\,U^{-1}\begin{pmatrix} 0 \\ \eta \end{pmatrix} = -g\begin{pmatrix} 0 \\ \eta \end{pmatrix}$$

(8.3)

Owing to (8.1) and (4.11), the equation (6.29) splits into a pair of uncoupled two component equations in ξ and η corresponding to *opposite eigenvalues* of the charge operator G [39], [40] :

$$\left[\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{s}\,.\nabla - i\frac{g}{hc}\left(W + \mathbf{s}\,.\mathbf{B}\right)\right]\xi = 0$$

$$\left[\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{s}\,.\nabla + i\frac{g}{hc}\left(W - \mathbf{s}\,.\mathbf{B}\right)\right]\eta = 0$$

(8.4)

They exchange between themselves by C, P, T transformations :

$$P : \mathbf{x} \to -\mathbf{x}\ ;\ t \to t\ ;\ \mathbf{B} \to \mathbf{B}\ ;\ W \to -W\ ;\ \xi \leftrightarrow \eta$$

$$T : \mathbf{x} \to \mathbf{x}\ ;\ t \to -t\ ;\ \mathbf{B} \to -\mathbf{B}\ ;\ W \to W\ ;\ \xi \leftrightarrow \eta$$

$$C : g \to g\ ;\ -i\,s_2\,\xi^* \to \eta;\ \ i\,s_2\,\eta^* \to \xi$$

(8.5)

They describe two *charge conjugated* particles — a *monopole* and an *antimonopole* — with the same charge constant but *opposite helicities* and going up and down the time.

The charge operator G is thus related to the helicity and our monopole appears as a kind of excited neutrino : the system (8.4) reduces to the neutrino two-component equations if $\mathsf{g} = 0$.

The eq. (8.4) are invariant under the gauge transformation (note the opposite signs of ϕ for ξ and η) :

$$\xi \to \exp\left(i\,\frac{\mathsf{g}}{hc}\,\phi\right)\,\xi \;;\; \eta \to \exp\left(-\,i\,\frac{\mathsf{g}}{hc}\,\phi\right)\,\eta \;;\; \mathsf{W} \to \mathsf{W} + \frac{1}{c}\frac{\partial\phi}{\partial t} \;;\; \mathbf{B} \to \mathbf{B} - \nabla\phi \tag{8.6}$$

9. Chiral currents.

The gauge law (8.6) entails, respectively, for the two equations (8.4), the *conservation of two chiral currents* :

$$\frac{1}{c}\frac{\partial\left(\xi^{+}\xi\right)}{\partial t} - \nabla\xi^{+}\mathbf{s}\,\xi = 0 \;;\; \frac{1}{c}\frac{\partial\eta^{+}\eta}{\partial t} + \nabla\eta^{+}\mathbf{s}\,\eta = 0 \tag{9.1}$$

These currents are thus defined as :

$$\mathsf{X}_{\mu} = \left\{\xi^{+}\xi,\, -\,\xi^{+}\mathbf{s}\,\xi\right\} \;;\; \mathsf{Y}_{\mu} = \left\{\eta^{+}\eta,\, \eta^{+}\mathbf{s}\,\eta\right\} \tag{9.2}$$

Its is easy to prove that they are *isotropic* :

$$\mathsf{X}_{\mu}\mathsf{X}_{\mu} = 0 \;;\; \mathsf{Y}_{\mu}\mathsf{Y}_{\mu} = 0 \tag{9.3}$$

They exchange between themselves by parity :

$$\mathsf{P} : \mathbf{x} \to -\,\mathbf{x} \;;\; \mathsf{X}_{\mu} \leftrightarrow \mathsf{Y}_{\mu} \tag{9.4}$$

which justifies their denomination as *chiral currents*.

Making use of eq. (8.1), we find the following decomposition of the polar and axial vectors defined in (6.11) :

$$\mathsf{J}_{\mu} = \mathsf{X}_{\mu} + \mathsf{Y}_{\mu} \;;\; \Sigma_{\mu} = \mathsf{X}_{\mu} - \mathsf{Y}_{\mu} \tag{9.5}$$

It seems to be a good idea to consider the chiral isotropic currents X_{μ} and Y_{μ} as the fundamental currents and to define the electric and magnetic currents as their sum and their difference (with suitable charge factors). Identities (6.17) are now easily proved because, using (6.11) and (8.1), we find :

$$\Omega_1 = \xi^{+}\eta + \eta^{+}\xi \;;\; \Omega_2 = i\left(\xi^{+}\eta - \eta^{+}\xi\right) \;;\; \rho^2 = 4\left(\xi^{+}\eta\right)\left(\eta^{+}\xi\right) \tag{9.6}$$

The fact that one of the currents (J_{μ} or Σ_{μ}) must be time-like and the other one space-like appears as a trivial property of the difference and the sum of two isotropic vectors. The space-like vector is J_{μ} because one can show that $\mathsf{X}_{\mu}\,\mathsf{Y}_{\mu} < 0$.

We understand that our magnetic current $\mathsf{K}_{\mu} = \mathsf{g}\Sigma_{\mu}$ may be space-like because the true magnetic currents are the conservative chiral currents $\mathsf{g}\mathsf{X}_{\mu}$ and $-\mathsf{g}\mathsf{Y}_{\mu}$ whereas K_{μ} is only their sum.

It is interesting to add a remark concerning the role of the chiral currents in the Dirac theory for the electron. In the Dirac equation, (9.1) cannot hold because we have no chiral invariance and we find instead the equalities :

$$\frac{1}{c}\frac{\partial(\xi^+\xi)}{\partial t} - \nabla\xi^+\mathbf{s}\,\xi = \frac{m_0 c}{h}\,\Omega_2 \; ; \; \frac{1}{c}\frac{\partial\eta^+\eta}{\partial t} + \nabla\eta^+\mathbf{s}\,\eta = -\frac{m_0 c}{h}\,\Omega_2$$

(9.7)

The second invariant Ω_2 appears as a source of magnetism in the Dirac eqation. The sum and the difference of these equations give the conservation of electricity and the Uhlenbeck and Laporte relation :

$$\partial_\mu J_\mu = 0 \; ; \; \partial_\mu \Sigma_\mu + 2\,\frac{m_0 c}{h}\,\Omega_2 = 0$$

(9.8)

10. The geometrical optics approximation and the monopole in an electric central field.

Now we must verify that we find the correct Poincaré equation and the Birkeland effect. Let us introduce in the first equation (8.4) the following expression of the spinor ξ :

$$\xi = a\,e^{iS/h}$$

(10.1)

where a is a two-component spinor and S a phase. At zeroth order in h, we have :

$$\left[\frac{1}{c}\left(\frac{\partial S}{\partial t} - g\,W\right) - \left(\nabla S + \frac{g}{c}\mathbf{B}\right).\,\mathbf{s}\right] a = 0$$

(10.2)

This is an homogeneous system with respect to a. A necessary condition for a non trivial solution is :

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} - g\,W\right)^2 - \left(\nabla S + \frac{g}{c}\mathbf{B}\right)^2 = 0$$

(10.3)

This is nothing but a relativistic Jacobi equation with zero mass and we may define the kinetic energy, the impulse and the linear Lagrange momentum :

$$E = -\frac{\partial S}{\partial t} + g\,W \; ; \; \mathbf{p} = \nabla S + \frac{g}{c}\mathbf{B} \; ; \; \mathbf{P} = \nabla S$$

(10.4)

The Hamiltonian function will be equal to :

$$H = c\sqrt{\left(\mathbf{P} + \frac{g}{c}\mathbf{B}\right)^2} - g\,W$$

(10.5)

and a classical calculation gives as equation of motion :

$$\frac{d\mathbf{p}}{dt} = g\left(\nabla W + \frac{\partial \mathbf{B}}{\partial t}\right) - \frac{g}{c}\,v \times \text{curl }\mathbf{B}$$

(10.6)

The eq. (4.13) gives the classical form :

$$\frac{d\,\mathbf{p}}{dt} = g\left(\mathbf{H} - 1/c\,\mathbf{v} \times \mathbf{E}\right) \tag{10.7}$$

But we must not forget that the mass of our particle is equal to zero, so that $\mathbf{v}$ is the velocity of light and we cannot write : $\mathbf{p} = m\,\mathbf{v}$. But the equality : $\mathbf{p} = E/c^2\,\mathbf{v}$ still holds when the energy E is a constant, which will be the case in a coulombian electric field. We then have :

$$\frac{d^2\mathbf{r}}{dt^2} = -\lambda\,\frac{1}{r^3}\,\frac{d\mathbf{r}}{dt} \times \mathbf{r}\ ;\quad \lambda = \frac{egc}{E} \tag{10.8}$$

This is exactly the Poincaré equation (3.2) with a minus sign because we have chosen the left monopole. If we had chosen the right monopole, i. e. the second equation (8.4), we would have found, with the same transformation (10.1), the following equation for $\mathbf{b}$:

$$\left[\frac{1}{c}\left(\frac{\partial S}{\partial t} + g\,W\right) - \left(\nabla S - \frac{g}{c}\,\mathbf{B}\right).\mathbf{s}\right]b = 0 \tag{10.9}$$

This new equation may be deduced from eq. (10.2), by applying the P and T transformations (8.5) but it could seem surprising that (10.9) cannot be deduced from the corresponding charge conjugation. The reason is that charge conjugation must be applied not to the spinor $\mathbf{a}$ in the eq. (10.2) but to the spinor ξ in (10.1), which gives :

$$g \rightarrow g\ ;\ -i\,s_2\,a^* \rightarrow b\ ;\ i\,s_2\,b^* \rightarrow a\ ;\ S \rightarrow -S \tag{10.10}$$

These equalities show, contrary to what a superficial comparison between (10.2) and (10.9) may suggest, that the latter is not deduced from the former by changing the sign of the magnetic charge but by changing the sign of the phase of the wave, with the same magnetic charge. Of course (10.9) gives the Poincaré equation (10.8) with a plus sign before λ.

11. The quantum description of a monopole in an electric central field. Angular eigenfunctions. Dirac's condition.

To solve the problem of a central field, we must introduce $W = 0$ and the expressions (5.15) or (5.16) of $\mathbf{B}$ in the chiral equations (8.4). First of all, a quite simple calculation gives the following integrals of motion, respectively for the monopole and the antimonopole or, equivalently for the left and the right monopole :

$$\mathbf{J}_\xi = h\left[\mathbf{r} \times \left(-i\,\nabla + D\mathbf{B}\right) + D\,\hat{\mathbf{r}} + \frac{1}{2}\,\mathbf{s}\right]$$
$$\mathbf{J}_\eta = h\left[\mathbf{r} \times \left(-i\,\nabla - D\mathbf{B}\right) - D\,\hat{\mathbf{r}} + \frac{1}{2}\,\mathbf{s}\right] \tag{11.1}$$

with the notations :

$$D = \frac{eg}{hc}\ ;\ \mathbf{B} = e\,B\ ;\ \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} \tag{11.2}$$

D is the *Dirac number* that already appeared in the *Dirac condition* (5.8). $\mathbf{J}_\xi$ and $\mathbf{J}_\eta$ only differ by the sign of D (i.e. by the sign of the eigenvalues of the charge operator). We shall restrict our study to the plus sign, that corresponds to the first equation (8.4) (the left monopole), and we shall drop the ξ index. It is easily shown that the components of **J** obey the relations of an angular momentum :

$$[J_2\,,\,J_3] = ih\ J_1\ \ ;\ [J_3\,,\,J_1] = ih\ J_2\ \ ;\ [J_1\,,\,J_2] = ih\ J_3 \tag{11.3}$$

Now, if we write **J** as :

$$\mathbf{J} = h\left[\ \Lambda + \frac{1}{2}\,\mathbf{s}\right]\ ;\ \ \Lambda = \ \mathbf{r} \times \left(-\,i\,\nabla + D\mathit{B}\right) + D\,\hat{\mathbf{r}} \tag{11.4}$$

we recognize that $h\Lambda$ is *the quantum form of the Poincaré integral* (3.4). **J** is the sum of this integral and of the spin operator : **J** is the *total quantum angular momentum* of the monopole in an electric coulombian field, the exact analogue of the corresponding classical quantity. Of course, the components of $h\Lambda$ obey the same relations (11.3) as the components of **J**.

In terms of polar angles, we find, from the definition (11.4) of **Λ** and the polar form (5.16) of **B** :

$$\Lambda^+ = \Lambda_1 + i\ \Lambda_2 = e^{i\varphi}\left(i\cot\theta\ \frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\theta} + \frac{D}{\sin\theta}\right)$$

$$\Lambda^- = \Lambda_1 - i\ \Lambda_2 = e^{-i\varphi}\left(i\cot\theta\ \frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\theta} + \frac{D}{\sin\theta}\right)$$

$$\Lambda_3 = -\,i\ \frac{\partial}{\partial\varphi} \tag{11.5}$$

It is interesting to note that, owing to our choice (5.15) for the electromagnetic gauge, there is no additional term in Λ_3 as it occurred with the Dirac solution (see for instance [42], [43]).

Now, we need the eigenstates Z(θ,φ) of Λ^2 and Λ_3, knowing from (11.3), that the eigenvalue equations are :

$$\Lambda^2 Z = j(j+1)\,Z\ \ ;\ \Lambda_3\,Z = \ m\,Z\ \ ;\ j = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 3,\ldots\ ;\ m = -\,j,\ -\,j+1,\ \ldots\,j-1, \tag{11.6}$$

In order to simplify the calculation of the functions Z(θ,φ), we shall introduce a new angle χ, the meaning of which will soon appear, and we consider the product :

$$\mathcal{D}\,(\theta,\varphi,\chi) = e^{iD\chi}\,Z\,(\theta,\varphi) \tag{11.7}$$

These functions are eigenstates of operators R_κ that are easily derived from (11.5) :

$$R^+ = R_1 + i\,R_2 = e^{i\varphi}\left(i\cot\theta\,\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\chi}\right)$$

$$R^- = R_1 - i\,R_2 = e^{-i\varphi}\left(i\cot\theta\,\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\chi}\right)$$

$$R_3 = -i\,\frac{\partial}{\partial\varphi}$$

(11.8)

Obviously, the eigenvalues are the same as those of Z :

$$R^2 D = j(j+1)\,D \ ; \ R_3 D = m\,D$$

(11.9)

The R_k are well-known : they are the *infinitesimal operators of the rotation group* written in the fixed referential. θ, φ, χ are the nutation, the precession and the proper rotation. The role of the rotation group is not surprising because of the *spherical symmetry* of the system constituted by a monopole in a central electric field.

Our eigenfunction problem is thus trivialy solved : instead of the cumbersome calculations of "monopole harmonics", we see, under the simple assumption of *continuity of the wave functions on the rotation group*, that the angular functions are the *generalized spherical functions*, i.e. the matrix elements of the irreducible unitary representations of the rotation group [39], [40], [44], [45]. These functions are also the eigenfunctions of the spherical top. This coincidence was quoted by Tamm [46] without explanation, but here, the explanation is evident because we already know the analogy between a symmetrical top and a monopole in a central field.

The eigenstates of R^2 and R_3 are (see any textbook on group theory) :

$$D_j^{m',m}(\theta,\varphi,\chi) = e^{i(m\varphi + m'\chi)}\,d_j^{m',m}(\theta)$$

(11.10)

$$d_j^{m',m}(\theta) = N\,(1-u)^{-(m-m')/2}\,(1-u)^{-(m-m')/2}(1+u)^{-(m+m')/2}$$
$$\times \left(\frac{d}{du}\right)^{j-m}\left[(1-u)^{(j-m')/2}(1+u)^{(j+m')/2}\right]$$

(11.11)

$$u = \cos\theta \ ; \ N = \frac{(-1)^{j-m}(i)^{m-m'}}{2^j}$$

(11.12)

$$j = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 3,\dots \ ; \ m, m' = -j,\ -j+1,\ \dots\ j-1$$

(11.13)

The normalization factor N is taken from group theory : the representation matrix is unitary and normed to unity. If we want normed eigenfunctions and real $d(\theta)$ functions in (11.11), we must take as "monopole harmonics" (eigenfunctions of Λ) :

$$Z_j^{m',m}(\theta,\varphi) = \sqrt{2j+1}\ D_j^{m',m}(\theta,\varphi,0)\,(i)^{m'-m}$$

(11.14)

The proper rotation angle χ disappears because the monopole was implicitly supposed to be *punctual*, contrary to the symmetric top that has a spatial extension. Nevertheless, there is a projection, different from zero, of the orbital angular momentum on the symmetry axis, due to the chirality of the magnetic charge. *The eigenvalue associated to this projection is the quantum number* m'.

This is a crucial point. If we compare (11.7) and (11.10), we see that the quantum number m' is nothing but the Dirac number D. This means that the continuity of the wave function on the rotation group implies the quantization of the Dirac number D :

$$D = m' \tag{11.15}$$

But, in virtue of the relations (11.13), we have :

$$m' = 0, \frac{1}{2}, 1, \frac{3}{2}, 3, \dots = \frac{n}{2} \tag{11.16}$$

Taking into account the definition (11.2) of D, the equality (11.15) is thus identical to the Dirac condition (5.8) and the latter appears as a consequence of the spherical symmetry of the system and of the continuity on the rotation group. As we have already announced it, the factor "one half" has nothing to do with strings beginning at the origin of electric charge : it is a consequence of the double connexity of the rotation group. Let us quote, concerning these questions, an interesting work of T.W. Barrett in which the role of the rotation group in electromagnetic field theories is extensively developed [47].

12. Radial functions.

For a monopole, the harmonics with spin are [40] :

$$\Omega_j^{m',m}(+) = \begin{pmatrix} \sqrt{\dfrac{j+m}{2j+1}} \quad Z_j^{m',m-1} \\ \sqrt{\dfrac{j-m+1}{2j+1}} \quad Z_j^{m',m} \end{pmatrix}$$

$$\Omega_j^{m',m}(-) = \begin{pmatrix} \sqrt{\dfrac{j-m+1}{2j+1}} \quad Z_j^{m',m-1} \\ -\sqrt{\dfrac{j+m}{2j+1}} \quad Z_j^{m',m} \end{pmatrix} \tag{12.1}$$

They correspond respectively to the eigenvalues $k = j \pm 1/2$ of the total angular momentum $\mathbf{J}$. For $k = j - 1/2$, we have :

$$\mathbf{J}^2 \, \Omega_{j-1}^{m',m}(+) = k(k+1) \, \Omega_{j-1}^{m',m}(+) \; ; \; \mathbf{J}^2 \, \Omega_j^{m',m}(-) = k(k+1) \, \Omega_j^{m',m}(-) \tag{12.2}$$

The solution of the radial problem consists in introducing the following expansion in the first equation (8.4) and to find the radial functions $F^{\pm}(r)$:

$$\xi = e^{-i\omega t} \left[F_{j-1}^+(r) \, \Omega_{j-1}^{m',m}(+) + F_j^-(r) \, \Omega_j^{m',m}(-) \right] \tag{12.3}$$

We shall not give the technical details, that may be found in reference [40], except for two points :

1) One needs, for the calculation, the following formulae :

$$\mathbf{s}.\hat{\mathbf{r}}\ \Omega_{j-1}^{m',m}(+) = \cos\Theta'\ \Omega_{j-1}^{m',m}(+) + \sin\Theta'\ \Omega_j^{m',m}(-)$$

$$\mathbf{s}.\hat{\mathbf{r}}\ \Omega_j^{m',m}(-) = \sin\Theta'\ \Omega_{j-1}^{m',m}(+) - \cos\Theta'\ \Omega_j^{m',m}(-) \tag{12.4}$$

$$\cos\Theta' = \frac{m'}{j} = \frac{D}{j}\ ;\ \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} \tag{12.5}$$

It is worth to note that the angle Θ' that appears is these formulae is the half-angle of the Poincaré cone Fig. 3, generated by the precession of the symmetry axis around the orbital angular momentum.

2) Denoting F(r) and G(r) as:

$$F = \begin{pmatrix} F_{j-1}^+ \\ F_j^- \end{pmatrix}\ ;\ G = \begin{pmatrix} G_{j-1}^+ \\ G_j^- \end{pmatrix}\ ;\ F = \frac{1}{2}\exp\left[i\,s_2\left(\frac{\pi}{4} - \frac{\Theta'}{2}\right)\right]G \tag{12.6}$$

the result of the calculation is :

$$G = \sqrt{\frac{\omega}{c}r}\begin{pmatrix} i\,J_{l-1/2}\left(\frac{\omega}{c}r\right) \\ J_{l+1/2}\left(\frac{\omega}{c}r\right) \end{pmatrix}\ ;\ l = j\sin\Theta' = \sqrt{j^2 - m'^2} \tag{12.7}$$

The important point is that ω is not quantized : the monopole in a coulombian electric field is always in a ionizing state. This fact, predicted by Dirac, may be guessed for two reasons : 1) Because we know the *spiraling motion* on the cone described in the classical case by Poincaré and we know that our equation has the Poincaré equation as a classical limit. 2) The potential **B** given in (5.15) has an infinite string and thus, the wave equation cannot have square integrable solutions.

The function (12.7) are also the massless limit of the solutions of the problem of an electrically charged fermion in the field of an infinitely massive monopole [23], which is obvious for formal symmetry reasons. But it must be stressed that the essential difference between our theory and *all the others* is the presence of the *charge operator* G = g γ_5 that modifies the chiral properties of the monopole and the question of charge conjugation.

13. A massive monopole.

Until now, our monopole was massless, which is not surprising in a gauge theory, but our equation (6.28) is unique only under the assumption of linearity. There are other possible equations that are chiral gauge invariant but they are non linear. A chiral invariant term must be independent from the angle A and it may be proved [39], [40] that the most general form is a function of ρ (defined by (6.14)). Thus we have the following lagrangian (where *M* is an arbitrary function) :

$$L = \frac{1}{2}\overline{\psi}\gamma_\mu\left[\partial_\mu\right]\psi - \frac{g}{hc}\overline{\psi}\gamma_\mu\gamma_5 B_\mu\psi + \frac{1}{4}\frac{M(\rho^2)c}{h}\psi \tag{13.1}$$

from which we find a general equation with a non linear term of mass depending on an arbitrary function *m* (the derivative of *M*) :

$$\left[\gamma_\mu\left(\partial_\mu - \frac{g}{hc}\,\gamma_5\,B_\mu\right)\psi + \frac{1}{2}\frac{m(\rho^2)\,c}{h}\,(\Omega_1 - i\,\Omega_2\,\gamma_5)\right]\psi = 0 \tag{13.2}$$

By definition, this equation is chiral invariant and the magnetic current given in (7.1) is conserved. Applying (7.2), one can see that the equation is P — invariant but not by C or T, only by CT, the so called "weak time reversal" :

$$CT: x_k \to x_k \;;\; x_4 \to -x_4 \;;\; B_k \to -B_k \;;\; B_4 \to B_4 \;;\; g \to g \;;\; \psi \to \gamma_3\gamma_1\psi^* \tag{13.3}$$

In the Weyl representation (8.1), eq. (13.2) is equivalent to the system :

$$\frac{1}{c}\frac{\partial\xi}{\partial t} - \mathbf{s}\,.\nabla\xi - i\frac{g}{hc}(W + \mathbf{s}\,.\mathbf{B})\,\xi + i\frac{m\left(4\left|\xi^+\eta\right|^2\right)c}{h}\,(\eta^+\xi)\,\eta = 0$$

$$\frac{1}{c}\frac{\partial\eta}{\partial t} + \mathbf{s}\,.\nabla\eta + i\frac{g}{hc}(W - \mathbf{s}\,.\mathbf{B})\,\eta + i\frac{m\left(4\left|\xi^+\eta\right|^2\right)c}{h}\,(\xi^+\eta)\,\xi = 0 \tag{13.4}$$

In virtue of (8.5), this system is P and CT — invariant :

$$CT: \mathbf{x} \to \mathbf{x}\;;\; t \to -t\;;\; \mathbf{B} \to -\mathbf{B}\;;\; W \to W\;;\; g \to g\;;$$
$$- i\,s_2\,\xi^* \to \eta;\;\; i\,s_2\,\eta^* \to \xi \tag{13.5}$$

In general, eqs. (13.4) are coupled, contrary to (8.4). The non linear mass term introduces a coupling, but not very strong. The isotropic chiral currents (see (9.1), (9.2)) are still separately conserved and the coupling vanishes when :

$$\rho = 2\left|\xi^+\eta\right| = 0 \tag{13.6}$$

This obviously happens when $\xi = 0$ or $\eta = 0$, which corresponds to one of the chiral components of the linear monopole, but it also happens in a more interesting case :

$$\xi = f(\mathbf{x},t)\,s_2\,\eta^* \;\Rightarrow\; \xi = e^{i\,\theta(\mathbf{x},t)}\,s_2\,\eta^* \tag{13.7}$$

where $f(\mathbf{x},t)$ is an arbitrary function that may be reduced to a phase term $\exp i\,\theta(\mathbf{x},t)$ under the assumption that both components ξ and η are separately normalized.

Up to the θ phase factor, the condition (13.7) is a generalization of the Majorana condition and it was shown that, *under this condition, there are monopole solutions that split into two components, even in the Dirac equation with a linear term of mass, where m_0 is the ordinary rest mass* [48] :

$$\frac{1}{c}\frac{\partial \xi}{\partial t} - \mathbf{s} . \nabla \xi - i \frac{g}{hc}\left(W + \mathbf{s} .\mathbf{B}\right)\xi + i \frac{m_0 c}{h} e^{i\theta} s_2 \xi^* = 0$$

$$\frac{1}{c}\frac{\partial \eta}{\partial t} + \mathbf{s} . \nabla \eta + i \frac{g}{hc}\left(W - \mathbf{s} .\mathbf{B}\right)\eta + i \frac{m_0 c}{h} e^{i\theta} s_2 \eta^* = 0$$

$$(13.8)$$

We shall not develope this question and we go back to the problem with $m_0 = 0$. Suppose that, in (13.7), $\theta = 0$: this means that chiral components are charge conjugated, they constitute a *pair monopole — anti-monopole* ; the definition (9.2) shows that the chiral currents are thus equal :

$$X_\mu = Y_\mu$$

$$(13.9)$$

There is no more chirality and, from (9.5), we have :

$$J_\mu = 2 X_\mu \; ; \; K_\mu = 0$$

$$(13.10)$$

In other words : 1) The electric current becomes isotropic, but this is not interesting for us, because we have supposed that the particle has no electric charge ; 2) More interesting is the fact that *the magnetic current vanishes for a pair of charge conjugated monopoles, although their charges are not of opposite sign.*

If we were living in an aether made of such pairs of monopoles, it would be very difficult to observe them, which means that perhaps the apparent rarity of monopoles is not due to a true rarity, but to the difficulty of observing only one of them. It must not be forgotten that we are living in a world full of electrons and that it is not so easy to "see" one of them !

14. The presence of tachyons in the non linear monopole equation.

It is easy to see that the phases of ξ and η are independent in eq. (13.4). We can introduce two plane waves with constant spinors a and b and two different phases :

$$\xi = a \, e^{\,i\left(\omega t - \mathbf{k}.\mathbf{r}\right)} \; ; \; \eta = b \, e^{\,i\left(\omega' t - \mathbf{k}'.\mathbf{r}\right)}$$

$$(14.1)$$

in the equations without external field. We find :

$$\left(\frac{\omega}{c} + \mathbf{s} .\mathbf{k}\right)a + \frac{m\left(|a^+b|\right)c}{h}(b^+a)\,b = 0$$

$$\left(\frac{\omega'}{c} - \mathbf{s} .\mathbf{k}'\right)b + \frac{m\left(|a^+b|\right)c}{h}(a^+b)\,a = 0$$

$$(14.2)$$

Multiplying the first equation by $(\omega'/c — \mathbf{s}.\mathbf{k}')$ and using the second equation, we find a linear system, with respect to the spinor a :

$$\left(\Omega + \mathbf{s}.\mathbf{K} - \frac{M^2 c^2}{h^2}\right)a = 0$$

$$(14.3)$$

With :

$$\Omega = \left(\frac{\omega \omega'}{c^2} - \mathbf{k} . \mathbf{k'} \right) \ ; \ \mathbf{K} = \frac{1}{c} \left(\omega' \, \mathbf{k} - \omega \, \mathbf{k'} \right) + i \, \mathbf{k} \times \mathbf{k'} \ ; \ M = |a^+b| \times m \left(|a^+b| \right)$$

$$(14.4)$$

Now, in order to have non trivial solutions of eq. (14.3) we must equal to zero the determinant. This gives a dispersion relation [40]:

$$\left(\frac{\omega^2}{c^2} - k^2 \right) \left(\frac{\omega'^2}{c^2} - k'^2 \right) - 2 \left(\frac{\omega \omega'}{c^2} - \mathbf{k} . \mathbf{k'} \right) \frac{M^2 c^2}{h^2} + \frac{M^4 c^4}{h^4} = 0$$

$$(14.5)$$

In virtue of the last eq. (14.4), *M* depends on a and b, except if the term of mass, in the system (13.4), has the particular form :

$$m \left(|\xi^+ \eta| \right) = \frac{m_0}{|\xi^+ \eta|} \quad (m_0 = \text{Const}) \ \Rightarrow \ \boxed{M = m_0}$$

$$(14.6)$$

It is interesting that, in this case, eqs. (13.4) are homogeneous in ξ and η, which are normalizable. There are reasons to believe that such an equation may have physical interest even for an electrically charged particle [49], [50].

With a general function *M*, two kinds of waves (14.1) are particularly interesting :

1) $\omega = \omega'$, $\mathbf{k} = \mathbf{k'}$: both monopoles have the same phase and the dispersion relation reduces to :

$$\frac{\omega^2}{c^2} = k^2 + \frac{M^2 c^2}{h^2}$$

$$(14.7)$$

This is the ordinary dispersion relation of a massive particle, say a *bradyon*, adopting the terminology of the tachyon theory because our second case is :

2) $\omega = - \omega'$, $\mathbf{k} = - \mathbf{k'}$: the phases are of opposite signs and the dispersion relation becomes :

$$\frac{\omega^2}{c^2} = k^2 - \frac{M^2 c^2}{h^2}$$

$$(14.8)$$

This is the dispersion relation of a supraluminal particle, a *tachyon*, the theory of which was extensively developed for many years, in particlular by E. Recami and coworkers [51], [52], [53]. The wave equation (13.4) seems to be the first one in which tachyon solutions appear, although it was originally written for quite independent reasons.

This non linear equation was more accurately described in some works in view of finding soliton solutions and stability properties [54], [55]. Nevertheless, the following result is unpublished :

Consider the nonlinear equations (13.4) in the case of a coulombian electric field, that is with a pseudo-potential (5.15) or (5.16). The equations cannot be separated as were the equations (8.4) and the 2×2 matrix operators (11.1) cannot represent the total angular momentum. The corresponding operator is the 4×4 operator :

$$\mathbf{J} = h \left[\mathbf{r} \times \left(- i \, \nabla + \gamma_4 \, DB \right) + \gamma_4 \, D \, \hat{\mathbf{r}} + \frac{1}{2} \mathbf{S} \right] \ ; \ \mathbf{S} = \begin{pmatrix} \mathbf{s} & 0 \\ 0 & \mathbf{s} \end{pmatrix}$$

$$(14.9)$$

that combines the preceding 2×2 operators. **J** commutes with the Hamiltonian of the whole linear system (8.4) ; it is an integral of motion. But it would be meaningless to look for a "commutation with the

nonlinear Hamiltonian" of the system (13.4), in order to prove that **J** is also an integral of the nonlinear system.

We must go back to the definition of an integral of motion in quantum mechanics and directly verify that the *mean value* of the operator **J** is a constant in virtue of the wave equations (13.4). If ψ is a solution of eq. (13.4), we find indeed :

$$\frac{\partial}{\partial t} \int \psi^{+} \, \mathbf{J} \, \psi \, dx \, dy \, dz = 0 \; ; \; \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

(14.10)

which confirms that the nonlinear system (13.4) has the same constant angular momentum as the linear system (8.4).

15. Miscellaneous remarks.

1) A geometric property. When $m \, (\rho^2)$ is constant in eq. (13.2) and (13.4), it was shown (see [12], [40]) using an older work of Rodichev [56]), that the presence of a monopole may be considered as a *local torsion* of an affine twisted space, the total curvature of which is : $R = \text{Const.} \times \rho^2$. Therefore, an aether made of pairs of monopoles is a flat space (because in this case, $\rho = 0$) and the question of the observability of monopoles may be expressed in a geometrical form : if such an aether do exist, what must be done in order that a local torsion appear ?

2) A possible relation between magnetic monopoles and weak interactions.

We know that the equations (6.28) or (8.4) give the neutrino equations as a particular case, for $g = 0$, and g is quantized by the Dirac condition (5.8) (which is also a consequence of our equations) :

$$g = n \, g_0 \; ; \; g_0 = \frac{hc}{2e}$$

(15.1)

Therefore, this monopole may be considered as a "magnetically excited" neutrino. More exactly, we have a family of monopoles with different values of n and the neutrino is the *ground state* with $n = 0$. It is thus natural to ask the question : *is it possible that such monopoles have not only electromagnetic but weak interactions ?* And this question leads to another one : *is it possible to produce monopoles in weak reactions instead of neutrinos ?* If it is so, there must be different families of monopoles associated with the different leptons e, μ and τ.

Finally, this leads to the hypothesis that these monopoles could play a role in the magnetic activity of the sun, in particular in the *sunspots*. Apart from the neutrinos produced by weak interactions responsible for the solar energy, massless monopoles could appear and, contrary to the ordinary neutrinos, they would undergo an important loss of energy in the condensed matter and *it could be a possible explanation for the lack of registered solar neutrinos*.

3) The Cerenkov radiation of a monopole.

Assuming the above hypothesis holds true, it is interesting to ask the following question : if massless monopoles are emitted in condensed matter by a weak reaction or, if some of these hypothetical monopoles created in the sun were able to reach the earth, would it be possible to oberve an emission of light ? An obvious idea is the Cerenkov radiation. It is not difficult apply the classical theory of Tamm and Franck to this problem. We shall not give the calculation, but only the result that seems the most interesting :

— In the classical Cerenkov radiation emitted by an electric charge, we have in the direction Oz of the propagation of the wave :

$$E_z \,{}^{\circ}\, 0 \;\; ; \;\; H_z = 0 \qquad\qquad (15.2)$$

— In the case of a magnetic charge, we have on the contrary :

$$E_z = 0 \;\; ; \;\; H_z \,{}^{\circ}\, 0 \qquad\qquad (15.3)$$

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